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Clément Dombry, Z. Kabluchko

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Clément Dombry[1] and Zakhar Kabluchko[2]

October 9, 2014

Abstract

With any max-stable random process η on $\mathcal{X} = \mathbb{Z}^d$ or $\mathbb{R}^d$, we associate a random tessellation of the parameter space $\mathcal{X}$. The construction relies on the Poisson point process representation of the max-stable process η which is seen as the pointwise maximum of a random collection of functions $\Phi = \{\phi_i, i \geq 1\}$. The tessellation is constructed as follows: two points $x, y \in \mathcal{X}$ are in the same cell if and only if there exists a function $\phi \in \Phi$ that realizes the maximum η at both points x and y, i.e. $\phi(x) = \eta(x)$ and $\phi(y) = \eta(y)$.

We characterize the distribution of cells in terms of coverage and inclusion probabilities. Most interesting is the stationary case where the asymptotic properties of the cells are strongly related to the ergodic properties of the non-singular flow generating the max-stable process. For example, we show that: i) the cells are bounded almost surely if and only if η is generated by a dissipative flow; ii) the cells have positive asymptotic density almost surely if and only if η is generated by a positive flow. We also provide a simple correspondence between the ergodic/mixing properties of the max-stable random field η and the geometry of the cells.

Key words: max-stable random field, random tessellation, non-singular flow representation, ergodic properties.
AMS Subject classification. Primary: 60G70 Secondary: 60D05, 60G52, 60G60, 60G55, 60G10, 37A10, 37A25.

[1]Université de Franche-Comté, Laboratoire de Mathématiques de Besançon, UMR CNRS 6623, 16 route de Gray, 25030 Besançon cedex, France. Email: clement.dombry@univ-fcomte.fr

[2]Institute of Stochastics, Ulm University, Helmholtzstraße 18, 89069 Ulm, Germany. Email: zakhar.kabluchko@uni-ulm.de

1 Introduction

Max-stable random fields provide popular and meaningful models for spatial extremes. The reason is that they appear as the only possible non-degenerate limits for normalized pointwise maxima of independent and identically distributed random fields. The one-dimensional marginal distributions of max-stable fields belong to the parametric class of Generalized Extreme Value distributions. Being interested mostly in the dependence structure, we will restrict our attention to max-stable fields with standard unit Fréchet margins. A max-stable random field $\eta = (\eta(x))_{x \in \mathcal{X}}$ on $\mathcal{X} \subset \mathbb{R}^d$ is then defined by the following properties:

- max-stability:

$$n^{-1} \bigvee_{i=1}^{n} \eta_i \stackrel{d}{=} \eta \quad \text{for all } n \geq 1,$$

  where $(\eta_i)_{1 \leq i \leq n}$ are i.i.d. copies of η, $\bigvee$ is the pointwise maximum, and $\stackrel{d}{=}$ denotes the equality of finite-dimensional distributions;

- unit Fréchet margins:

$$\mathbb{P}[\eta(x) \leq u] = \exp(-1/u) \quad \text{for all } x \in \mathcal{X} \text{ and } u > 0.$$

A fundamental tool in the study of max-stable processes is their spectral representation (see e.g. de Haan [1], Giné *et al.* [6], Penrose [14]): any stochastically continuous max-stable process η can be written in the form

$$\eta(x) = \bigvee_{i \geq 1} U_i Y_i(x), \quad x \in \mathcal{X}, \tag{1}$$

where

- $(U_i)_{i \geq 1}$ is the decreasing enumeration of the points of a Poisson point process on $(0, +\infty)$ with intensity $u^{-2} \mathrm{d}u$,
- $(Y_i)_{i \geq 1}$ are i.i.d. copies of a non-negative stochastic process Y on $\mathcal{X}$ such that $\mathbb{E}[Y(x)] = 1$ for all $x \in \mathcal{X}$,
- the sequences $(U_i)_{i \geq 1}$ and $(Y_i)_{i \geq 1}$ are independent.

In this paper, we focus on max-stable random fields defined on $\mathcal{X} = \mathbb{Z}^d$ or $\mathbb{R}^d$. In the case $\mathcal{X} = \mathbb{R}^d$ we always assume that η has continuous sample paths. Equivalently, the spectral process Y has continuous sample paths and

$$\mathbb{E}\Big[\sup_{x \in K} Y(x)\Big] < \infty \text{ for every compact set } K \subset \mathbb{R}^d. \tag{2}$$

Representation (1) has a nice interpretation pointed out by Smith [20] and Schlather [19]. In the context of a rainfall model, we can interpret each index $i \geq 1$ as a *storm event*, where U_i stands for the intensity of the storm and Y_i stands for its shape; then $U_i Y_i(x)$ represents the amount of precipitation due to the storm event i at point $x \in \mathcal{X}$, and $\eta(x)$ is the maximal precipitation over all storm events at this point. This interpretation raises a natural question: what is the shape of the region $C_i \subset \mathcal{X}$ where the storm i is extremal? More formally, we define the *cell* associated to the storm event $i \geq 1$ by

$$C_i = \{x \in \mathcal{X};\ U_i Y_i(x) = \eta(x)\}, \quad i \geq 1.$$

It is a (possibly empty) *random closed subset* of $\mathcal{X}$. Note that each point $x \in \mathcal{X}$ belongs almost surely to a unique cell (the point process $\{U_i Y_i(x)\}_{i \geq 1}$ is a Poisson point process with intensity $u^{-2}\mathrm{d}u$ so that the maximum $\eta(x)$ is almost surely attained uniquely). In the discrete setting $\mathcal{X} = \mathbb{Z}^d$, the cells $(C_i)_{i \geq 1}$ are almost surely pairwise disjoint and they cover $\mathbb{Z}^d$; in the continuous setting $\mathcal{X} = \mathbb{R}^d$, the cells $(C_i)_{i \geq 1}$ form a random covering of $\mathbb{R}^d$ by closed sets with disjoint interiors. We call $(C_i)_{i \geq 1}$ the *random tessellation* of $\mathcal{X}$ associated with η. Let us stress that in this paper the terms *cell* and *tessellation* are meant in a broader sense than in stochastic geometry where they originated. Here, a cell is a general (not necessarily convex or connected) random closed set and a tessellation is a random covering by closed sets.

A drawback of this approach is that the distribution of the cell C_i depends on the specific representation (1) and in particular on the ordering of the points $(U_i)_{i \geq 1}$. For instance, with the convention that the sequence $(U_i)_{i \geq 1}$ is decreasing, the cell C_1 is stochastically larger than the other cells. To avoid this, we introduce a canonical way to define the tessellation. The idea is that given a point $x \in \mathcal{X}$, there is almost surely a unique storm event giving the maximum precipitation at this point. Then, the cell $C(x)$ is exactly the set of points where this particular storm is maximal. The formal definition is as follows.

Definition 1. *For $x \in \mathcal{X}$, the cell of x is the random closed subset*

$$C(x) = \{y \in \mathcal{X};\ \exists i \geq 1,\ U_i Y_i(x) = \eta(x) \text{ and } U_i Y_i(y) = \eta(y)\}. \quad (3)$$

The cell $C(x)$ is non-empty since it contains x. In the case $\mathcal{X} = \mathbb{Z}^d$, for any two points $x_1, x_2 \in \mathbb{Z}^d$, the cells $C(x_1)$ and $C(x_2)$ are almost surely either equal or disjoint. In the case $\mathcal{X} = \mathbb{R}^d$, for any two points $x_1, x_2 \in \mathbb{R}^d$, the cells $C(x_1)$ and $C(x_2)$ are almost surely either equal or have disjoint interiors.

The purpose of this paper is to study some properties of the tessellation $(C(x))_{x \in \mathcal{X}}$. The following lemma provides a first simple but important observation.

Lemma 2. *The distribution of the tessellation* $(C(x))_{x\in\mathcal{X}}$ *depends on the distribution of the max-stable process* η *only and not on the specific representation* (1).

To prove the lemma, introduce the functional point process (which will play a key role in the sequel)

$$\Phi = \{\phi_i,\ i \geq 1\} \quad \text{where } \phi_i = U_i Y_i,\ i \geq 1.$$

Note that ϕ_i are elements of $\mathcal{F}_0 = \mathcal{F}(\mathcal{X}, [0,+\infty)) \setminus \{0\}$, the set of non-negative and continuous functions on $\mathcal{X}$ excluding the zero function. (We may assume without loss of generality that Y does not vanish identically). The set $\mathcal{F}_0$ is endowed with the σ-algebra generated by the coordinate mappings. It is well known (see, e.g., de Haan and Ferreira [2]) that Φ is a Poisson point process on $\mathcal{F}_0$ with intensity measure μ given by

$$\mu(A) = \int_0^\infty \mathbb{P}[uY \in A] u^{-2} \mathrm{d}u, \quad A \subset \mathcal{F}_0 \text{ Borel}. \tag{4}$$

The measure μ is called the exponent measure or max-Lévy measure and is related to the multivariate cumulative distribution functions of η by

$$\begin{aligned} &\mathbb{P}[\eta(x_j) \leq z_j, j = 1, \ldots, n] \\ &\quad = \exp\left(-\mu(\{f \in \mathcal{F}_0;\ f(x_j) > z_j \text{ for some } j = 1, \ldots, n\})\right) \end{aligned}$$

for all $n \geq 1$, $x_1, \ldots, x_n \in \mathcal{X}$ and $z_1, \ldots, z_n > 0$. In particular, this shows that μ depends on the distribution of η only and does not depend on the specific representation (1). Now, Lemma 2 follows easily as the tessellation $(C(x))_{x\in\mathcal{X}}$ is a functional of the Poisson point process Φ with intensity μ.

The aim of this paper is to study some properties of the tessellation $(C(x))_{x\in\mathcal{X}}$ and to relate them to the properties of the max-stable random field $(\eta(x))_{x\in\mathcal{X}}$. The paper is structured as follows. In Section 2, we study the law of the cell $C(x)$ and provide some formulas for the inclusion and coverage probabilities as well as some examples. In Section 3, we focus on the stationary case and establish strong connections between asymptotic properties of $C(0)$ and ergodic properties of the non-singular flow associated with η. Theorem 12 relates the boundedness of the cell to the conservative/dissipative decomposition. Theorem 14 links the asymptotic density of the cell with the positive/null decomposition. We exhibit also strong relationships between the ergodic and mixing properties of η and the geometry of the cell $C(0)$. Proofs are collected in Sections 4 and 5. Some background as well as new results on non-singular flow representations of max-stable processes are postponed to an appendix.

2 Basic properties and examples

2.1 Basic properties

Our first result is a simple characterization of the distribution of the cell $C(x)$.

Theorem 3. *Consider a sample continuous max-stable random field η given by representation* (1)*. For every $x \in \mathcal{X}$ and every measurable set $K \subset \mathcal{X}$,*

$$\mathbb{P}[K \subset C(x)] = \mathbb{E}\left[\inf_{y \in K \cup \{x\}} \frac{Y(y)}{\eta(y)}\right] \tag{5}$$

and

$$\mathbb{P}[C(x) \subset K] = \mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)} - \sup_{y \in K^c} \frac{Y(y)}{\eta(y)}\right)^+\right]. \tag{6}$$

where Y is independent of η, $K^c = \mathcal{X} \setminus K$ is the complement of the set K, and $(z)^+ = \max(z, 0)$ is the positive part of z.

It is well known that the distribution of a random closed set $C \subset \mathcal{X}$ is completely determined by its capacity functional

$$\mathcal{X}_C(K) = \mathbb{P}[C \cap K \neq \emptyset], \quad K \subset \mathcal{X} \text{ compact},$$

see, e.g., Molchanov [13, Chapter 1]. Clearly, Theorem 3 implies that the capacity functional of the cell $C(x)$ is given by

$$\mathcal{X}_{C(x)}(K) \quad = \quad 1 - \mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)} - \sup_{y \in K} \frac{Y(y)}{\eta(y)}\right)^+\right].$$

Remark 4. It is worth noting that Weintraub [26] introduced (with a different terminology) the probability that two points x and y are in the same cell as a measure of dependence between $\eta(x)$ and $\eta(y)$. More precisely, he considered

$$\beta(x,y) = \mathbb{P}[y \in C(x)] = \mathbb{E}\left[\frac{Y(x)}{\eta(x)} \wedge \frac{Y(y)}{\eta(y)}\right] \quad x, y \in \mathcal{X}.$$

Clearly, $\beta(x,y) \in [0,1]$. One can prove easily that $\beta(x,y) = 0$ holds if and only if $\eta(x)$ and $\eta(y)$ are independent, while $\beta(x,y) = 1$ if and only if $\eta(x) = \eta(y)$ almost surely. Moreover, $\beta(x,y)$ can be compared to the extremal coefficient $\theta(x,y)$ which is another well-known measure of dependence for max-stable processes defined by

$$\theta(x,y) = -\log \mathbb{P}[\eta(x) \vee \eta(y) \leq 1] \in [1,2]. \tag{7}$$

According to Stoev [21, Proposition 5.1], we have

$$\frac{1}{2}(2 - \theta(x,y)) \leq \beta(x,y) \leq 2(2 - \theta(x,y)). \tag{8}$$

As a by-product of Theorem 3, we can provide an explicit expression for the mean volume of the cells. Denote by λ the discrete counting measure when $\mathcal{X} = \mathbb{Z}^d$ or the Lebesgue measure when $\mathcal{X} = \mathbb{R}^d$. The volume of $C(x)$ is defined by $\mathrm{Vol}(C(x)) = \lambda(C(x))$. In the discrete case, $\mathrm{Vol}(C(x))$ is the cardinality of $C(x)$.

Corollary 5. *The cell $C(x)$ has expected volume*

$$\mathbb{E}[\mathrm{Vol}(C(x))] = \int_{\mathcal{X}} \mathbb{E}\left[\frac{Y(x)}{\eta(x)} \wedge \frac{Y(y)}{\eta(y)}\right] \lambda(\mathrm{d}y).$$

In particular, Equation (8) implies that the cell $C(x)$ has finite expected volume if and only if $\int_{\mathcal{X}}(2 - \theta(x, y))\lambda(\mathrm{d}y) < +\infty$. Another consequence of Theorem 3 is an expression for the probability that the cell $C(x)$ is bounded.

Corollary 6. *Let $x \in \mathcal{X}$. The cell $C(x)$ is bounded with probability*

$$\mathbb{P}[C(x) \text{ bounded}] = \mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)} - \limsup_{y\to\infty} \frac{Y(y)}{\eta(y)}\right)^+\right].$$

Furthermore, the following statements are equivalent:

i) the cell $C(x)$ is bounded a.s.;

ii) as $y \to \infty$, $\frac{Y(y)}{\eta(y)} \to 0$ a.e. on the event $\{Y(x) \neq 0\}$.

Remark 7. In the case when the max-stable process η is stationary, we will see in Section 3.2 below that condition ii) can be replaced by the following one: $Y(y) \to 0$ a.s. as $y \to \infty$.

2.2 Examples

As an illustration and to get some intuition, we provide some simulations of max-stable processes together with the associated tessellations.

Example 8. The isotropic Smith process is defined by

$$\eta(x) = \bigvee_{i\geq 1} U_i h(x - X_i), \quad x \in \mathbb{R}^d,$$

where $\{(U_i, X_i), i \geq 1\}$ is a Poisson point process on $(0, \infty) \times \mathbb{R}^d$ with intensity $u^{-2}\mathrm{d}u\mathrm{d}x$ and $h(x) = (2\pi)^{-d/2} \exp(-\|x\|^2/2)$ is the standard Gaussian d-variate density function. The Smith process is a stationary max-stable process that belongs to the class of *moving maximum processes* and is hence mixing. Surprisingly, the associated tessellation

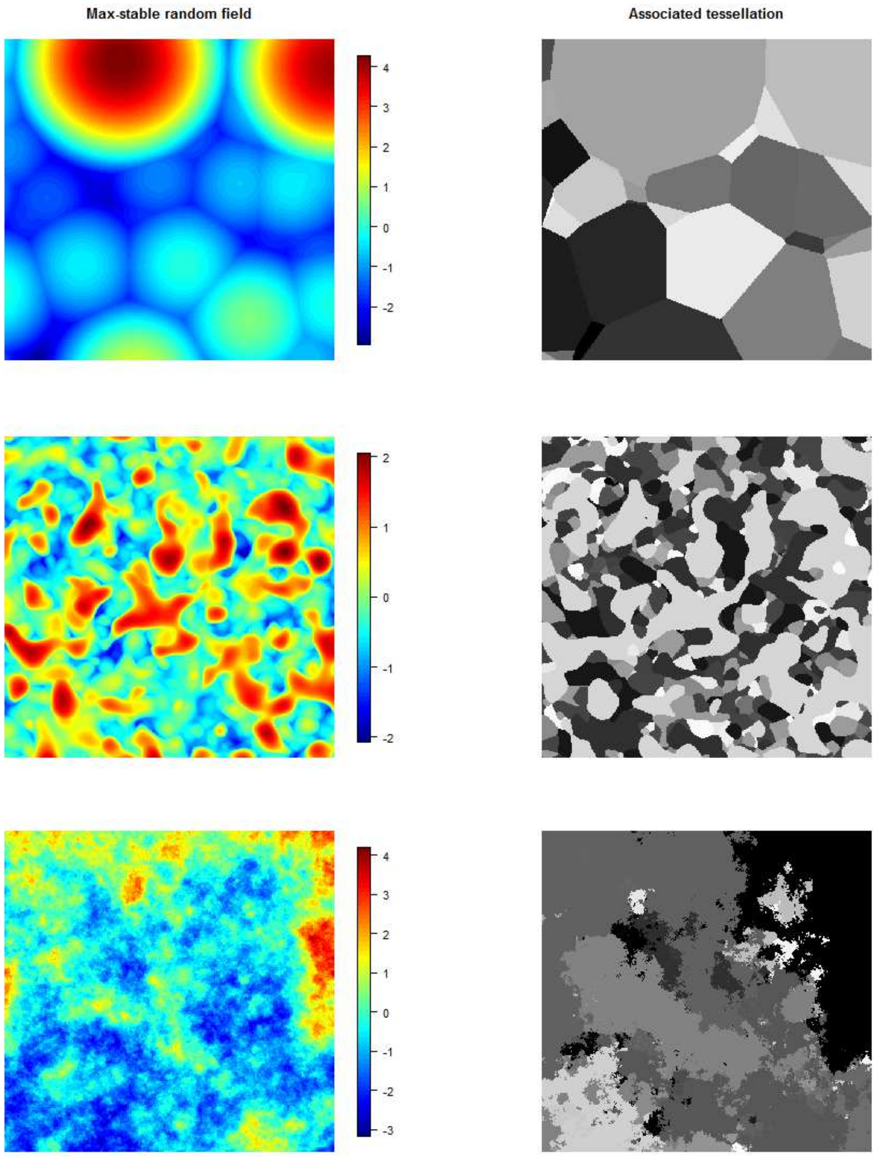

Figure 1: Realizations of a max-stable random field on $\mathcal{X} = [-5, 5]^2$ (left column) and of the associated tessellation (right column) for various models. Top: the Smith model with Gaussian shape function (Example 8). Middle: the extremal Gaussian model with Gaussian correlation $\rho(h) = \exp(-\|h\|^2/2)$ (Example 9). Bottom: the Brown–Resnick model with variogram $\gamma(h) = 2\|h\|$ (Example 10). To obtain a better contrast, max-stable random fields are plotted with Gumbel margins.

is exactly the so-called *Laguerre tessellation* studied in great detail by Lautensack and Zuyev [12]. Indeed, the cell C_i is given by

$$C_i = \{x \in \mathbb{R}^d;\ \|x - X_i\|^2 - 2\ln(U_i) \le \|x - X_j\|^2 - 2\ln(U_j),\quad j \neq i\}.$$

The cells are convex bounded polygons as can be seen in the first line of Figure 2.2.

Example 9. The stationary Gaussian extremal process originally introduced by Schlather [19] corresponds to the case when the spectral process Y in representation (1) is given by

$$Y(x) = \sqrt{\frac{\pi}{2}} \max(W(x), 0), \quad x \in \mathcal{X},$$

where W is a stationary Gaussian process on $\mathcal{X}$ with zero mean, unit variance and correlation function $\rho(h) = \mathbb{E}[W(0)W(h)]$, $h \in \mathcal{X}$. The associated extremal coefficient is given by

$$\theta(h) = 2T_2\left[\sqrt{\frac{2}{1-\rho(h)^2}} - \sqrt{\frac{1-\rho(h)^2}{2}}\rho(h)\right], \quad h \in \mathcal{X},$$

where T_2 is the cumulative distribution function of a Student distribution with 2 degrees of freedom. Typically, $\rho(h) \to 0$ as $h \to \infty$, so that $\theta(h) \to 2T_2(\sqrt{2}) < 2$ and η is neither mixing nor ergodic (see Stoev [21] or Kabluchko and Schlather [8]). Equation (8) entails that

$$\liminf_{h\to\infty} \mathbb{P}[h \in C(0)] > 0$$

suggesting that the cells are not bounded which is consistent with the simulation on the second line of Figure 2.2. Note also the very particular shape of the cells which are neither convex nor connected. Still, they have a smooth boundary due to the particular choice of the correlation function $\rho(h) = \exp(-\|h\|^2/2)$ that yields smooth Gaussian sample paths.

Example 10. Brown–Resnick processes [9] form a flexible class of max-stable processes. They are given by Equation (1) with the spectral process of the form

$$Y(x) = \exp\left(W(x) - \frac{1}{2}\sigma^2(x)\right), \quad x \in \mathcal{X},$$

where W is a stationary increment centered Gaussian process, and $\sigma^2(x) = \operatorname{Var} W(x)$. Surprisingly, the process η is stationary [9]. Its distribution is completely characterized by the variogram

$$\gamma(h) = \operatorname{Var}(W(x+h) - W(x)).$$

The extremal coefficient function is given by

$$\theta(h) = 2\Phi\left(\frac{1}{2}\sqrt{\gamma(h)}\right), \quad h \in \mathcal{X}.$$

Typically, $\gamma(h) \to \infty$ as $h \to \infty$, so that $\theta(h) \to 2$ and η is mixing. Equation (8) entails that

$$\lim_{h\to\infty} \mathbb{P}[h \in C(0)] = 0.$$

From the asymptotics $1 - \Phi(u) \sim 1/(\sqrt{2\pi}u)\, e^{-u^2/2}$, $u \to +\infty$, for the normal tail function and from Corollary 5 it follows that that the cell $C(0)$ has finite expected volume provided that the following condition is satisfied:

$$\liminf_{h\to\infty} \frac{\gamma(h)}{\log \|h\|} > 8d.$$

We conjecture that the cell $C(0)$ is a.s. bounded if the same condition holds with $4d$ on the right-hand side and that the constant $4d$ is sharp.

We can see on the third line of Figure 2.2 that the cells have a very rough shape, due to the particular choice of the variogram $\gamma(h) = 2\|h\|$ that yields rough Gaussian paths.

3 The stationary case: asymptotic properties of cells

3.1 Stationary max-stable random fields

In the sequel, we focus on the case when η is a *stationary*, sample continuous max-stable random field on $\mathcal{X} = \mathbb{Z}^d$ or $\mathbb{R}^d$. The structure of stationary max-stable processes was first investigated by de Haan and Pickands [3]. Recently, further results were obtained by exploiting the analogy between the theory of max-stable and sum-stable processes. Inspired by the works of Rosinski and Samorodnitsky [15, 17, 16, 18], the representation theory of stationary max-stable random fields via non singular flows was developed independently by Kabluchko [7], Wang and Stoev [25] and Wang *et al.* [24]. See also Kabluchko and Stoev [10] for an extension to sum- and max-infinitely divisible processes. In these works, the conservative/dissipative and positive/null decompositions of the non-singular flow play a major role.

To avoid technical details of non-singular ergodic theory, we use a naive approach based on cone decompositions of max-stable processes (see, for example, Wang and Stoev [25, Theorem 5.2]). The links between this approach and the non-singular ergodic theory are explored in the Appendix.

The following simple lemma about the cone decompositions of max-stable processes will be useful. Recall that $\mathcal{F}_0 = \mathcal{F}(\mathcal{X}, [0, +\infty)) \setminus \{0\}$ denotes the set of continuous, non-negative functions on $\mathcal{X}$ excluding the zero function. A measurable subset $\mathcal{C} \subset \mathcal{F}_0$ is called a *cone* if for all $f \in \mathcal{C}$ and $u > 0$, $uf \in \mathcal{C}$. The cone $\mathcal{C}$ is said to be *shift-invariant* if for all $f \in \mathcal{C}$ and $x \in \mathcal{X}$, we have $f(\cdot + x) \in \mathcal{C}$.

Lemma 11. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two measurable, shift-invariant cones such that $\mathcal{F}_0 = \mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$. Let η be a stationary max-stable process given by representation* (1)*. Consider the decomposition $\eta = \eta_1 \vee \eta_2$ with*

$$\eta_1(x) = \bigvee_{i \geq 1} U_i Y_i(x) 1_{\{Y_i \in \mathcal{C}_1\}} \quad \textit{and} \quad \eta_2(x) = \bigvee_{i \geq 1} U_i Y_i(x) 1_{\{Y_i \in \mathcal{C}_2\}}.$$

Then, η_1 and η_2 are stationary and independent max-stable processes whose distribution depends only on the distribution of η and not on the specific representation (1)*.*

3.2 Boundedness of cells

We will prove that the boundedness of the cell $C(x)$ is strongly connected with the conservative/dissipative decomposition of the max-stable process η. Introduce the following shift-invariant cones of functions:

$$\mathcal{F}_C = \left\{ f \in \mathcal{F}_0;\ \limsup_{x \to \infty} f(x) > 0 \right\}, \tag{9}$$

$$\mathcal{F}_D = \left\{ f \in \mathcal{F}_0;\ \lim_{x \to \infty} f(x) = 0 \right\}. \tag{10}$$

The conservative/dissipative decomposition of η is given by

$$\eta_C(x) = \bigvee_{i \geq 1} U_i Y_i(x) 1_{Y_i \in \mathcal{F}_C}, \tag{11}$$

$$\eta_D(x) = \bigvee_{i \geq 1} U_i Y_i(x) 1_{Y_i \in \mathcal{F}_D}. \tag{12}$$

According to Lemma 11, the processes η_C and η_D are independent stationary max-stable processes such that

$$\eta(x) = \eta_C(x) \vee \eta_D(x), \quad x \in \mathcal{X}.$$

For an interpretation of η_C and η_D in terms of the conservative/dissipative decomposition of the non-singular flow generating η, we refer the reader to Appendix A.2. The following theorem relates this conservative/dissipative decomposition to the boundedness of the cell $C(x)$.

Theorem 12. *Let $x \in \mathcal{X}$. The following events are equal modulo null sets:*

$$\{C(x) \text{ is unbounded}\} = \{\eta_C(x) > \eta_D(x)\}, \tag{13}$$

$$\{C(x) \text{ is bounded}\} = \{\eta_D(x) > \eta_C(x)\}. \tag{14}$$

We denote by α_C and α_D the scale parameters of the 1-Fréchet random variables $\eta_C(x)$ and $\eta_D(x)$ respectively, i.e. for all $z > 0$,

$$\mathbb{P}[\eta_C(x) \leq z] = \exp(-\alpha_C/z), \quad \mathbb{P}[\eta_D(x) \leq z] = \exp(-\alpha_D/z). \tag{15}$$

Note that $\alpha_D + \alpha_C = 1$ and that α_C and α_D do not depend on $x \in \mathcal{X}$. We say that η is purely conservative (resp. purely dissipative) if $\alpha_D = 0$ (resp. $\alpha_C = 0$).

Corollary 13. *Let $x \in \mathcal{X}$. We have:*

i) $\mathbb{P}[C(x)$ *is unbounded*$] = \alpha_C$*,*

ii) $\mathbb{P}[C(x)$ *is bounded*$] = \alpha_D$*,*

iii) $C(x)$ *is unbounded a.s. if and only if* η *is purely conservative,*

iv) $C(x)$ *is bounded a.s. if and only if* η *is purely dissipative.*

3.3 Asymptotic density of cells

Next we consider the decomposition of η into positive and null components and relate it to the asymptotic density of the cell $C(x)$. For this purpose, we introduce a new construction of the positive/null decomposition of max-stable processes which simplifies and extends to the dimension $d \geq 1$ the construction from Samorodnitsky [18] and Wang and Stoev [25, Example 5.4].

For $r > 0$, we write $B_r = [-r, r]^d \cap \mathcal{X}$. We equip $\mathcal{X}$ with a measure λ which is either the counting or the Lebesgue measure, when $\mathcal{X} = \mathbb{Z}^d$ or $\mathcal{X} = \mathbb{R}^d$, respectively. Consider the shift-invariant cones of functions

$$\mathcal{F}_P = \left\{ f \in \mathcal{F}_0;\ \lim_{r \to \infty} \frac{1}{\lambda(B_r)} \int_{B_r} f(x)\lambda(\mathrm{d}x) > 0 \right\}, \tag{16}$$

$$\mathcal{F}_N = \left\{ f \in \mathcal{F}_0;\ \liminf_{r \to \infty} \frac{1}{\lambda(B_r)} \int_{B_r} f(x)\lambda(\mathrm{d}x) = 0 \right\}. \tag{17}$$

In the definition of $\mathcal{F}_P$, we assume that the limit exists. The stationarity of η implies that $Y \in \mathcal{F}_P \cup \mathcal{F}_N$ a.s.; see Theorem 36 in Appendix A.3. According to Lemma 11, the corresponding decomposition is

$$\eta_P(x) = \bigvee_{i \geq 1} U_i Y_i(x) 1_{Y_i \in \mathcal{F}_P}, \tag{18}$$

$$\eta_N(x) = \bigvee_{i \geq 1} U_i Y_i(x) 1_{Y_i \in \mathcal{F}_N}, \tag{19}$$

where the processes η_N and η_P are independent, stationary, max-stable, and

$$\eta(x) = \eta_P(x) \vee \eta_N(x), \quad x \in \mathcal{X}.$$

This is the so-called positive/null decomposition; see Appendix A.3 for more details.

Given a measurable subset $C \subset \mathcal{X}$, we define its lower and upper asymptotic densities by

$$\delta^-(C) = \liminf_{r \to +\infty} \frac{\lambda(C \cap B_r)}{\lambda(B_r)}, \quad \delta^+(C) = \limsup_{r \to +\infty} \frac{\lambda(C \cap B_r)}{\lambda(B_r)}.$$

If $\delta^-(C) = \delta^+(C)$, the common value is called the asymptotic density of C and denoted by $\delta(C)$. The following theorem relates the positive/null decomposition of η to the asymptotic density of the cell $C(x)$.

Theorem 14. *Let $x \in \mathcal{X}$. The following events are equal modulo null sets:*

$$\{\delta(C(x)) > 0\} = \{\eta_P(x) > \eta_N(x)\}, \tag{20}$$

$$\{\delta^-(C(x)) = 0\} = \{\eta_N(x) > \eta_P(x)\}, \tag{21}$$

where the notation $\delta(C(x)) > 0$ means that the asymptotic density $\delta(C(x))$ exists and is positive.

We denote by α_P and α_N the scale parameters of the 1-Fréchet random variables $\eta_P(x)$ and $\eta_N(x)$ respectively, i.e. for all $z > 0$,

$$\mathbb{P}[\eta_P(x) \leq z] = \exp(-\alpha_P/z) \quad \text{and} \quad \mathbb{P}[\eta_N(x) \leq z] = \exp(-\alpha_N/z).$$

Note that $\alpha_P + \alpha_N = 1$ and that α_P and α_N do not depend on x. We say that the max-stable process η is generated by a positive (resp. null) flow if $\alpha_N = 0$ (resp. $\alpha_P = 0$).

Corollary 15. *Let $x \in \mathcal{X}$. We have:*

i) $\mathbb{P}[\delta(C(x)) > 0] = \alpha_P$.

ii) $\mathbb{P}[\delta^-(C(x)) = 0] = \alpha_N$.

iii) $\delta(C(x)) > 0$ *a.s. if and only if η is generated by a positive flow.*

iv) $\delta^-(C(x)) = 0$ *a.s. if and only if η is generated by a null flow.*

3.4 Connection with ergodic properties

Ergodic and mixing properties of max-stable random fields have been studied intensively by Stoev [21, 22], Wang *et al.* [24] and Kabluchko and Schlather [8]. A major result is that a max-stable process is ergodic if and only if it is generated by a null flow. Also, a simple characterization using the extremal coefficient is known:

- η is ergodic if and only if $\theta(h) \to 2$ in Cesàro mean as $h \to \infty$;
- η is mixing if and only if $\theta(h) \to 2$ as $h \to \infty$.

Interestingly, these results can be reinterpreted in terms of the geometric properties of the tessellation.

Proposition 16. *Let η be a stationary, sample continuous max-stable random field on $\mathcal{X} = \mathbb{Z}^d$ or $\mathbb{R}^d$.*

1. *The following statements are equivalent:*
   (1.a) η *is ergodic.*
   (1.b) $\lim_{r\to+\infty} \mathbb{E}\left[\frac{\lambda(C(0)\cap B_r)}{\lambda(B_r)}\right] = 0.$
2. *The following statements are equivalent:*
   (2.a) η *is mixing.*
   (2.b) $\lim_{x\to\infty} \mathbb{P}[x \in C(0)] = 0.$

Next we focus on strong mixing properties of max-stable processes, see Dombry and Eyi-Minko [4]. The β-mixing coefficients of the random process η are defined as follows: for disjoint subsets $S_1, S_2 \subset \mathcal{X}$, we define

$$\beta(S_1, S_2) = \sup\left\{|\mathcal{P}_{S_1\cup S_2}(C) - (\mathcal{P}_{S_1} \otimes \mathcal{P}_{S_2})(C)|;\ C \in \mathcal{B}_{S_1\cup S_2}\right\}, \quad (22)$$

where $\mathcal{P}_S$ is the distribution (on the space $\mathbb{R}_+^S$) of the restriction of η to the set S, and $\mathcal{B}_S$ is the product σ-algebra on the space $\mathbb{R}_+^S$. For fixed $S \subset \mathcal{X}$ and $r > 0$, we write

$$\beta_r(S) = \beta(S, S_r^c) \text{ with } S_r^c = \{x \in S; d(x,S) \geq r\}.$$

We say that η is β-mixing if for all compact sets $S \subset \mathcal{X}$,

$$\lim_{r\to+\infty} \beta_r(S) = 0.$$

Proposition 17. *If η is a stationary max-stable random field such that $C(0)$ is almost surely bounded, then η is β-mixing.*

According to Corollary 13, $C(0)$ is a.s. bounded if and only if η is generated by a dissipative flow. So, Propostion 17 states that purely dissipative max-stable processes are β-mixing. We conjecture that the converse implication is also true:

Conjecture 18. *If η is a β-mixing stationary max-stable random field, then η is generated by a dissipative flow.*

We were not able to prove the conjecture, mainly because we lack a lower bound for the β-mixing coefficient $\beta(S_1, S_2)$ (only an upper bound is given in [4]).

4 Proofs related to section 2

4.1 Proof of Theorem 3

Proof of Theorem 3. We first prove Equation (5). For $f, g : \mathcal{X} \to \mathbb{R}$ and $K \subset \mathcal{X}$, we use the notation

$$f >_K g \quad \text{if and only if} \quad f(x) > g(x) \text{ for all } x \in K.$$

For $i \geq 1$, we define the random functions

$$\phi_i = U_i Y_i \quad \text{and} \quad m_i = \bigvee_{j \neq i} \phi_j.$$

Fix some $x \in \mathcal{X}$. Note that $x \in C_i$ if and only if $\phi_i(x) \geq m_i(x)$, whence (modulo sets of probability 0)

$$\begin{aligned} & \{K \subset C(x)\} \\ = \;& \{\exists i \geq 1,\ \phi_i(x) > m_i(x) \text{ and } \forall y \in K,\ \phi_i(y) > m_i(y)\} \\ = \;& \{\exists i \geq 1,\ \phi_i >_{K \cup \{x\}} m_i\}. \end{aligned}$$

The events $\{\phi_i >_{K\cup\{x\}} m_i\}$, $i \geq 1$, are pairwise disjoint so that

$$1_{\{K \subset C(x)\}} = \sum_{i \geq 1} 1_{\{\phi_i >_{K \cup \{x\}} m_i\}} \quad \text{a.s.}$$

Hence, we obtain

$$\mathbb{P}[K \subset C(x)] = \mathbb{E}\Big[\sum_{i \geq 1} 1_{\{\phi_i >_{K \cup \{x\}} m_i\}}\Big].$$

This expectation can be computed thanks to the Slivniak–Mecke formula (see, e.g., Stoyan *et al.* [23]). Recall that Φ is a Poisson point process with intensity μ and that m_i is a functional of $\Phi \setminus \{\phi_i\}$. The Slivniak–Mecke formula implies that

$$\mathbb{P}[K \subset C(x)] \;=\; \int_{\mathcal{F}_0} \mathbb{E}\left[1_{\{f >_{K \cup \{x\}} \eta\}}\right] \mu(\mathrm{d}f).$$

Using Equation (4), we compute

$$\begin{aligned}
\int_{\mathcal{F}_0} \mathbb{E}\left[1_{\{f >_{K\cup\{x\}} \eta\}}\right] \mu(\mathrm{d}f) &= \int_0^\infty \mathbb{E}\left[1_{\{uY >_{K\cup\{x\}} \eta\}}\right] u^{-2}\mathrm{d}u \\
&= \mathbb{E}\left[\int_0^\infty 1_{\{u > \sup_{K\cup\{x\}} \eta/Y\}} u^{-2}\mathrm{d}u\right] \\
&= \mathbb{E}\left[\inf_{K\cup\{x\}} Y/\eta\right].
\end{aligned}$$

This proves Equation (5).

The proof of Equation (6) relies on the same method and we give only the main ideas. We have (modulo sets of probability 0)

$$\{C(x) \subset K\} = \{\exists i \geq 1,\ \phi_i(x) > m_i(x) \text{ and } \phi_i <_{K^c} m_i\}$$

and

$$\mathbb{P}[C(x) \subset K] = \mathbb{E}\Big[\sum_{i\geq 1} 1_{\{\phi_i(x) > m_i(x)\}} 1_{\{\phi_i <_{K^c} m_i\}}\Big].$$

Slivniak–Mecke formula and Equation (4) entail that

$$\begin{aligned}
\mathbb{P}[C(x) \subset K] &= \int_{\mathcal{F}_0} \mathbb{E}\left[1_{\{f(x) > \eta(x)\}} 1_{\{f <_{K^c} \eta\}}\right] \mu(\mathrm{d}f) \\
&= \int_0^\infty \mathbb{E}\left[1_{\{uY(x) > \eta(x)\}} 1_{\{uY <_{K^c} \eta\}}\right] u^{-2}\mathrm{d}u.
\end{aligned}$$

Integrating with respect to du, we obtain

$$\begin{aligned}
\int_0^\infty 1_{\{uY(x) > \eta(x)\}} 1_{\{uY <_{K^c} \eta\}} u^{-2}\mathrm{d}u &= \int_0^\infty 1_{\{\eta(x)/Y(x) < u < \inf_{K^c} \eta/Y\}} u^{-2}\mathrm{d}u \\
&= \left(Y(x)/\eta(x) - \sup_{K^c} Y/\eta\right)^+,
\end{aligned}$$

whence Equation (6) follows. □

4.2 Proof of Corollaries 5 and 6

Proof of Corollary 5. By Fubini's Theorem, the expected volume of the cell $C(x)$ is equal to

$$\mathbb{E}[\mathrm{Vol}(C(x))] = \mathbb{E}\left[\int_{\mathcal{X}} 1_{\{y \in C(x)\}}\right] = \int_{\mathcal{X}} \mathbb{P}[y \in C(x)] \lambda(\mathrm{d}y)$$

and, according to Theorem 3,

$$\mathbb{P}[y \in C(x)] = \mathbb{E}\left[\frac{Y(x)}{\eta(x)} \wedge \frac{Y(y)}{\eta(y)}\right].$$

□

Proof of Corollary 6. For $n \geq 1$, we use the notation $B_n = [-n, n]^d \cap \mathcal{X}$. The sequence of events $\{C(x) \subset B_n\}$, $n \geq 1$, is non-decreasing and we have

$$\{C(x) \text{ bounded}\} = \bigcup_{n \geq 1} \{C(x) \subset B_n\},$$

whence

$$\mathbb{P}[C(x) \text{ bounded}] = \lim_{n \to \infty} \mathbb{P}[C(x) \subset B_n].$$

Using Equation (6), we get

$$\mathbb{P}[C(x) \subset B_n] = \mathbb{E}\Big[\Big(Y(x)/\eta(x) - \sup_{B_n^c} Y/\eta\Big)^+\Big].$$

As $n \to +\infty$, the sequence $\sup_{B_n^c} Y/\eta$ decreases to $\limsup_\infty Y/\eta$. The monotone convergence theorem entails that

$$\lim_{n \to \infty} \mathbb{E}\Big[\Big(Y(x)/\eta(x) - \sup_{B_n^c} Y/\eta\Big)^+\Big] = \mathbb{E}\Big[\Big(Y(x)/\eta(x) - \limsup_{\infty} Y/\eta\Big)^+\Big],$$

whence we deduce

$$\mathbb{P}[C(x) \text{ bounded}] = \mathbb{E}\Big[\Big(Y(x)/\eta(x) - \limsup_{\infty} Y/\eta\Big)^+\Big].$$

In order to prove the equivalence of the statements (i) and (ii), we note that

$$0 \leq \left(Y(x)/\eta(x) - \limsup_{\infty} Y/\eta\right)^+ \leq Y(x)/\eta(x)$$

and $\mathbb{E}[Y(x)/\eta(x)] = 1$. The latter equality holds because $Y(x)$ is independent of $1/\eta(x) \sim \mathrm{Exp}(1)$. Note also that $(a-b)^+ = a$ (for $a, b \geq 0$) if and only if $a = 0$ or $b = 0$. Hence, the equality

$$\mathbb{E}\left[\left(Y(x)/\eta(x) - \limsup_{\infty} Y/\eta\right)^+\right] = 1$$

occurs if and only if $\limsup_\infty Y/\eta = 0$ a.e. on the event $\{Y(x) \neq 0\}$. This proves the equivalence of (i) and (ii). $\square$

5 Proofs related to section 3

Proof of Lemma 11. By the uniqueness of the max-Lévy measure, the max-stable process η is stationary if and only if its max-Lévy measure μ is stationary. By the properties of Poisson point processes, $\Phi \cap \mathcal{C}_i$, $i = 1, 2$, are independent Poisson point processes with intensity measures $\mathrm{d}\mu_i = 1_{\mathcal{C}_i} \mathrm{d}\mu$. The max-stable processes η_1 and η_2 are hence independent with exponent measures μ_1 and μ_2, respectively. Since the cone $\mathcal{C}_i$ is shift-invariant, so is the measure μ_i. Hence, the process η_i is stationary. Finally, the distribution of η_i is characterized by the max-Lévy measure $\mathrm{d}\mu_i = 1_{\mathcal{C}_i}\mathrm{d}\mu$ and does not depend on the representation (1). $\square$

5.1 Brown–Resnick stationary processes

The notion of Brown–Resnick stationarity introduced in Kabluchko *et al.* [9] will be useful.

Definition 19. *We say that the process $Y = (Y(x))_{x\in\mathcal{X}}$ is Brown–Resnick stationary if the associated max-stable process η defined by* (1) *is stationary.*

We will use the following lemma due to Kabluchko *et al.* [9, Corollary 8].

Lemma 20. *If Y and Y' are independent Brown–Resnick stationary processes, then YY' is also Brown–Resnick stationary.*

For future reference, we record the following by-product of Lemma 11 and its proof.

Lemma 21. *Let Y be a Brown–Resnick stationary process and C a shift-invariant cone, then $Y1_{\{Y\in C\}}$ is Brown–Resnick stationary.*

The next two lemmas are related to the conservative/dissipative decomposition of cones. Their proof is postponed to Appendix A.2. We recall that in this paper, in the continuous time case $\mathcal{X} = \mathbb{R}^d$, we focus on the setting when η and Y have continuous sample paths. Then we have for all compact sets $K \subset \mathcal{X}$,

$$\mathbb{P}\left[\sup_K \eta \leq u\right] = \exp\left(-\frac{1}{u}\mathbb{E}\left[\sup_K Y\right]\right), \quad u > 0.$$

with

$$\mathbb{E}\left[\sup_{x\in K} Y(x)\right] < \infty. \tag{23}$$

Note that in the discrete case $\mathcal{X} = \mathbb{Z}^d$, Equation (23) is trivially fulfilled because compact sets are finite.

Lemma 22. *Let Y be a (sample continuous) Brown–Resnick stationary process and let $K = [-1/2, 1/2]^d \cap \mathcal{X}$. Then, modulo null sets,*

$$\left\{\lim_{x\to\infty} Y(x) = 0\right\} = \left\{\int_{\mathcal{X}} \sup_{y\in K} Y(x+y)\lambda(\mathrm{d}x) < \infty\right\}.$$

In the case $\mathcal{X} = \mathbb{Z}^d$, the lemma takes the following simple form:

$$\left\{\lim_{x\to\infty} Y(x) = 0\right\} = \left\{\sum_{x\in\mathbb{Z}^d} Y(x) < \infty\right\} \quad \text{modulo null sets.}$$

For the next lemma, we need the notion of localizable cone.

Definition 23. *A shift-invariant cone $\mathcal{F}_L$ is said to be localizable if there exist mappings $L_1 : \mathcal{F}_L \to \mathcal{X}$ and $L_2 : \mathcal{F}_L \to (0, +\infty)$ such that for all $f \in \mathcal{F}_L$, $x \in \mathcal{X}$ and $u > 0$,*

- *$L_1(f(\cdot + x)) = L_1(f) - x$ and $L_1(uf) = L_1(f)$,*
- *$L_2(f(\cdot + x)) = L_2(f)$ and $L_2(uf) = uL_2(f)$.*

A typical example of localizable cone is the cone

$$\mathcal{F}_D = \left\{f \in \mathcal{F}_0; \lim_{\infty} f = 0\right\}.$$

In this case, a possible choice for the mappings L_1 and L_2 is

$$L_1(f) = \arg\max f \quad \text{and} \quad L_2(f) = \max f,$$

where $\arg\max f$ is the point $x \in \mathcal{X}$ achieving the maximum of $f(x)$ (if there are several such points, we take the smallest with respect to the lexicographic order).

Lemma 24. *Let Y be a (sample continuous) Brown–Resnick stationary process and let $\mathcal{F}_L$ be a localizable cone. Then,*

$$\{Y \in \mathcal{F}_L\} \subset \{Y \in \mathcal{F}_D\} \quad \textit{modulo null sets.}$$

5.2 Proofs of Theorem 12 and Corollary 13

In the next lemma, we gather some preliminary computations needed for the proof of Theorem 12.

Lemma 25. *We have:*

i) $\alpha_C = \mathbb{E}[Y(x)1_{\{Y\in\mathcal{F}_C\}}]$ *and* $\alpha_D = \mathbb{E}[Y(x)1_{\{Y\in\mathcal{F}_D\}}]$.

ii) $\mathbb{P}[\eta_C(x) > \eta_D(x)] = \alpha_C$ *and* $\mathbb{P}[\eta_D(x) > \eta_C(x)] = \alpha_D$.

iii) $\mathbb{P}[C(x)$ *bounded,* $\eta_C(x) > \eta_D(x)] = \mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)} - \limsup_\infty \frac{Y}{\eta}\right)^+ 1_{\{Y\in\mathcal{F}_C\}}\right]$.

iv) $\mathbb{P}[C(x)$ *bounded,* $\eta_D(x) > \eta_C(x)] = \mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)} - \limsup_\infty \frac{Y}{\eta}\right)^+ 1_{\{Y\in\mathcal{F}_D\}}\right]$.

Proof of Lemma 25. *i)* Recall that α_C and α_D were defined in (15). Using the definition of η_C, see (11), standard computations entail that

$$\begin{aligned}\mathbb{P}[\eta_C(x) \leq y] &= \mathbb{P}[\vee_{i\geq 1} U_i Y_i(x) 1_{\{Y_i \in \mathcal{F}_C\}} \leq y] \\ &= \exp\Big(- \int_0^\infty \mathbb{P}[uY(x)1_{\{Y\in\mathcal{F}_C\}} > y] u^{-2} \mathrm{d}u \Big) \\ &= \exp(-\mathbb{E}[Y(x)1_{\{Y\in\mathcal{F}_C\}}]/y),\end{aligned}$$

whence we deduce that $\alpha_C = \mathbb{E}[Y(x)1_{\{Y\in\mathcal{F}_C\}}]$. The formula for α_D is obtained in the same way. This proves statement i).

ii) The random variables $\eta_C(x)$ and $\eta_D(x)$ are independent and have Fréchet distribution with parameters α_C and α_D, respectively. Hence,

$$\begin{aligned}\mathbb{P}[\eta_C(x) > \eta_D(x)] &= \mathbb{E}[\exp(-\alpha_D/\eta_C(x))] \\ &= \int_0^{+\infty} \exp(-\alpha_D/u) \mathrm{d}(e^{-\alpha_C/u}) \\ &= \alpha_C.\end{aligned}$$

For the last equality, we use $\alpha_C + \alpha_D = 1$. Similarly,

$$\mathbb{P}[\eta_D(x) > \eta_C(x)] = \alpha_D$$

and statement ii) is proved.

iii) This statement is a variation of Corollary 6 and we give only the main lines of its proof. We first prove the following version of Equation (6): For all compact sets $K \subset \mathcal{X}$,

$$\begin{aligned}&\mathbb{P}[C(x) \subset K,\ \eta_C(x) > \eta_D(x)] \\ =\ &\mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)} - \sup_{y\in K^c} \frac{Y(y)}{\eta(y)} \right)^+ 1_{\{Y\in\mathcal{F}_C\}} \right]. \end{aligned} \tag{24}$$

Indeed, with the same notation as in the proof of Equation (6), we have

$$\begin{aligned}&\{C(x) \subset K,\ \eta_C(x) > \eta_D(x)\} \\ =\ &\{\exists i \geq 1,\ \phi_i(x) > m_i(x), \quad \phi_i <_{K^c} m_i \quad \text{and } \phi_i \in \mathcal{F}_C\}\end{aligned}$$

and the Slivnyak–Mecke formula entails that

$$\begin{aligned}&\mathbb{P}[C(x) \subset K,\ \eta_C(x) > \eta_D(x)] \\ =\ &\mathbb{E}\left[\sum_{i\geq 1} 1_{\{\phi_i(x) > m_i(x)\}} 1_{\{\phi_i <_{K^c} m_i\}} 1_{\{\phi_i \in \mathcal{F}_C\}} \right] \\ =\ &\int_{\mathcal{F}_0} \mathbb{E}\left[1_{\{f(x) > \eta(x)\}} 1_{\{f <_{K^c} \eta\}} 1_{\{f\in\mathcal{F}_C\}} \right] \mu(\mathrm{d}f).\end{aligned}$$

With similar computations as in the proof of Equation (6), Equation (24) is easily deduced. Then statement iii) follows from Equation (24) exactly in the same way as Corollary 6 follows from Equation (6).

iv) The proof of point iv) is similar and is omitted. □

Proof of Theorem 12. We first reduce the proof of Theorem 12 to the proof of the following two equations:

$$\mathbb{P}[C(x) \text{ bounded}, \ \eta_D(x) > \eta_C(x)] = \mathbb{P}[\eta_D(x) > \eta_C(x)] \tag{25}$$

and

$$\mathbb{P}[C(x) \text{ bounded}, \ \eta_C(x) > \eta_D(x)] = 0. \tag{26}$$

Indeed, Equation (25) implies the inclusion (modulo null sets)

$$\{\eta_D(x) > \eta_C(x)\} \subset \{C(x) \text{ bounded}\}.$$

Since $\{\eta_D(x) = \eta_C(x)\}$ is a null set, Equation (26) implies the reverse inclusion

$$\{C(x) \text{ bounded}\} \subset \{\eta_D(x) > \eta_C(x)\}.$$

We deduce that $\{C(x) \text{ bounded}\} = \{\eta_D(x) > \eta_C(x)\}$, thus proving Equation (14). Taking the complementary sets, we obtain Equation (13) since $\{\eta_D(x) = \eta_C(x)\}$ is a null set.

Proof of Equation (25) We first reduce the proof of Equation (25) to the proof of

$$\lim_{y\to\infty} \frac{Y(y)}{\eta(y)} 1_{\{Y\in\mathcal{F}_D\}} = 0 \quad \text{a.s.} \tag{27}$$

Indeed, Equation (27) and statements i), ii) and iv) of Lemma 25 entail that

$$\begin{aligned}
& \mathbb{P}[C(x) \text{ bounded}, \ \eta_D(x) > \eta_C(x)] \\
= \ & \mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)} - \limsup_{\infty} \frac{Y}{\eta}\right)^+ 1_{\{Y\in\mathcal{F}_D\}}\right] \\
= \ & \mathbb{E}\left[\frac{Y(x)}{\eta(x)} 1_{\{Y\in\mathcal{F}_D\}}\right] \\
= \ & \alpha_D \\
= \ & \mathbb{P}[\eta_D(x) > \eta_C(x)],
\end{aligned}$$

and we get Equation (25).

It remains to prove Equation (27). Note that the process $Y 1_{\{Y\in\mathcal{F}_D\}}$ is Brown–Resnick stationary. Lemma 22 implies that

$$\int_{\mathcal{X}} \sup_{y\in K} Y(x+y) 1_{\{Y\in\mathcal{F}_D\}} \lambda(\mathrm{d}x) < \infty \quad \text{a.s.}$$

On the other hand, let us consider the process $Z = \frac{Y}{\eta} 1_{\{Y \in \mathcal{F}_D\}}$. Since Y and η are Brown–Resnick stationary and the cone $\mathcal{F}_D$ is shift invariant, Lemmas 20 and 21 imply that $Z = \frac{Y}{\eta} 1_{\{Y \in \mathcal{F}_D\}}$ is Brown–Resnick stationary. Furthermore, for any compact set $K \subset \mathcal{X}$,

$$\begin{aligned}
& \mathbb{E}\left[\int_{\mathcal{X}} \sup_{y \in K} Z(x+y) \lambda(\mathrm{d}x) \;\Big|\; Y \right] \\
\leq\; & \mathbb{E}\left[\int_{\mathcal{X}} \frac{\sup_{y\in K} Y(x+y)}{\inf_{y \in K} \eta(x+y)} 1_{\{Y \in \mathcal{F}_D\}} \lambda(\mathrm{d}x) \;\Big|\; Y \right] \\
=\; & \mathbb{E}\left[\sup_{y \in K} \eta^{-1}(y) \right] \int_{\mathcal{X}} \sup_{y \in K} Y(x+y) 1_{\{Y \in \mathcal{F}_D\}} \lambda(\mathrm{d}x) < \infty \quad \text{a.s.}
\end{aligned}$$

In the last equation, we used the independence of Y and η, the stationarity of η and the fact that $\mathbb{E}\left[\sup_{y\in K} \eta^{-1}(y)\right] < \infty$ (see Dombry and Eyi Minko [4, Theorem 2.2]). As a consequence,

$$\int_{\mathcal{X}} \sup_{y \in K} Z(x+y) \lambda(\mathrm{d}x) < \infty \quad \text{a.s.}$$

and Lemma 22 implies that $\lim_{x\to\infty} Z(x) = 0$ a.s., thus proving Equation (27).

Proof of Equation (26) We consider the shift-invariant cone

$$\mathcal{F}_L = \left\{ f \in \mathcal{F}_0; \sup_{\mathcal{X}} f > \limsup_{\infty} f \right\}.$$

We will prove that the process $Z = \frac{Y}{\eta} 1_{\{Y \in \mathcal{F}_C\}}$ is Brown–Resnick stationary and satisfies

$$\mathbb{P}[Z \in \mathcal{F}_L] = 0. \tag{28}$$

After this has been done, Equation (26) can be deduced as follows. Equation (28) implies that

$$\frac{Y(x)}{\eta(x)} 1_{\{Y \in \mathcal{F}_C\}} \leq \sup_{\mathcal{X}} \left(\frac{Y}{\eta} 1_{\{Y \in \mathcal{F}_C\}} \right) \leq \left(\limsup_{\infty} \frac{Y}{\eta} 1_{\{Y \in \mathcal{F}_C\}} \right) \quad \text{a.s.,}$$

whence

$$\left(\frac{Y(x)}{\eta(x)} - \limsup_{\infty} \frac{Y}{\eta} \right)^+ 1_{\{Y \in \mathcal{F}_C\}} = 0 \quad \text{a.s.}$$

According to Lemma 25, statement iii), we obtain that

$$\begin{aligned}
& \mathbb{P}[C(x) \text{ bounded}, \; \eta_C(x) > \eta_D(x)] \\
=\; & \mathbb{E}\left[\left(\frac{Y(x)}{\eta(x)} - \limsup_{\infty} \frac{Y}{\eta} \right)^+ 1_{\{Y \in \mathcal{F}_C\}} \right] \\
=\; & 0,
\end{aligned}$$

and this proves Equation (26).

We now consider Equation (28). Clearly, Lemmas 20 and 21 imply that the process Z is Brown–Resnick stationary. Since the cone $\mathcal{F}_L$ is localizable (take $L_1(f) = \arg\max f$ and $L_2(f) = \max f$ in Definition 23), Lemma 24 entails that

$$\mathbb{P}[Z \in \mathcal{F}_L] \leq \mathbb{P}[Z \in \mathcal{F}_D].$$

So, it suffices to prove that $\mathbb{P}[Z \in \mathcal{F}_D] = 0$. Suppose by contradiction that $\mathbb{P}[Z \in \mathcal{F}_D] > 0$. Recalling that $Z = \frac{Y}{\eta} 1_{\{Y \in \mathcal{F}_C\}}$, we see that

$$\{Z \in \mathcal{F}_D\} = \{Y \in \mathcal{F}_C\} \cap \{Y/\eta \in \mathcal{F}_D\}.$$

On the set $\{Y \in \mathcal{F}_C\} = \{\limsup_\infty Y > 0\}$, one can construct a $\sigma(Y)$-measurable random sequence $x_n \to \infty$ such that

$$Y(x_n) \geq \frac{1}{2} \limsup_{\infty} Y > 0.$$

Then, on $\{Z \in \mathcal{F}_D\} \subset \{Y/\eta \in \mathcal{F}_D\} = \{\lim_\infty Y/\eta = 0\}$, we have necessarily $\eta(x_n) \to +\infty$. But η is stationary and independent of Y, so that $\eta(x_n)$ has a unit Fréchet distribution that does not depend on n. This leads to a contradiction and we must hence have $\mathbb{P}[Z \in \mathcal{F}_D] = 0$. This concludes the proof of Equation (28). □

Proof of Corollary 13. Theorem 12 and Lemma 25-ii) together yield

$$\mathbb{P}[C(x) \text{ unbounded}] = \mathbb{P}[\eta_C(x) > \eta_D(x)] = \alpha_C,$$

proving statement i). Statement ii) is proved similarly. Furthermore, η is purely dissipative if $\eta_C = 0$, which is equivalent to $\alpha_C = 0$. We deduce easily that η is purely dissipative if and only if $C(x)$ is bounded a.s. and this proves iii). The proof of iv) is similar. □

5.3 Proofs of Theorem 14 and Corollary 15

Proof of Theorem 14. It suffices to prove the following two inclusions (modulo null sets):

$$\{\eta_N(x) > \eta_P(x)\} \subset \{\delta^-(C(x)) = 0\} \tag{29}$$

and

$$\{\eta_P(x) > \eta_N(x)\} \subset \{\delta(C(x)) > 0\}. \tag{30}$$

Indeed, the events on the left-hand side are complementary, while the events on the right-hand side are disjoint. This implies that both (29) and (30) are, in fact, equalities modulo null sets.

Proof of Equation (29). Let us consider the cell of x with respect to the null component only. It is defined by

$$C_N(x) = \{y \in \mathcal{X};\ \exists i \geq 1,\ Y_i \in \mathcal{F}_N,\ U_iY_i(x) = \eta_N(x),\ U_iY_i(y) = \eta_N(y)\}.$$

Clearly, $\eta_N(x) > \eta_P(x)$ implies that $C(x) \subset C_N(x)$. We will prove that $\delta^-(C_N(x)) = 0$ on $\{\eta_N(x) > \eta_P(x)\}$ and this implies Equation (29).

We can suppose without loss of generality that $\eta = \eta_N$ is generated by a null flow and prove that the lower asymptotic density of $C(x) = C_N(x)$ is equal to zero. Kabluchko [7] and Wang *et al.* [24] proved that max-stable processes associated to null flows are ergodic. Hence, $\eta = \eta_N$ is ergodic. On the other hand, there is an alternative characterization of ergodicity in terms of the extremal coefficient (Stoev [22], Kabluchko and Schlather [8]): The process η is ergodic if and only if

$$\lim_{r\to+\infty} \frac{1}{\lambda(B_r)} \int_{B_r} (2 - \theta(0, y))\lambda(\mathrm{d}y) = 0, \tag{31}$$

where $\theta(x, y)$ is defined in Equation (7). In view of Equation (8), this is equivalent to

$$\lim_{r\to+\infty} \frac{1}{\lambda(B_r)} \int_{B_r} \beta(0, y)\lambda(\mathrm{d}y) = 0.$$

Since $\beta(0, y) = \mathbb{P}[y \in C(0)]$, we obtain that

$$\frac{1}{\lambda(B_r)} \int_{B_r} \beta(0, y)\lambda(\mathrm{d}y) = \mathbb{E}\Big[\frac{\lambda(C(0) \cap B_r)}{\lambda(B_r)}\Big],$$

whence Equation (31) is equivalent to

$$\lim_{r\to+\infty} \mathbb{E}\Big[\frac{\lambda(C(0) \cap B_r)}{\lambda(B_r)}\Big] = 0.$$

This implies the convergence in probability

$$\frac{\lambda(C(0) \cap B_r)}{\lambda(B_r)} \xrightarrow{\mathbb{P}} 0, \quad \text{as } r \to +\infty$$

and hence almost sure converge to 0 along a subsequence. We deduce that $\delta^-(C(0)) = 0$ almost surely and, by stationarity, the same holds true for $C(x)$, $x \in \mathcal{X}$. □

Proof of Equation (30). Possibly changing representation (1), we may suppose without loss of generality that the random processes $\tilde{Y}_i = Y_i 1_{\{Y_i \in P\}}$ are stationary; see Appendix A.3. We consider the cells

$$\tilde{C}_i = \{y \in \mathcal{X},\ U_i\tilde{Y}_i(y) = \eta(y)\}, \quad i \geq 1.$$

We will prove below that for every $i \geq 1$ with probability one,

$$\text{either } \delta(\tilde{C}_i) > 0 \text{ or } \lambda(\tilde{C}_i) = 0. \tag{32}$$

We show that this implies Equation (30). On the event $\{\eta_P(x) > \eta_N(x)\}$, there is a random index $i(x)$ such that $C(x) = \tilde{C}_{i(x)}$. Furthermore, since $x \in C(x)$, we have $\lambda(\tilde{C}_{i(x)}) > 0$ (this is clear in the discrete case, in the continuous case, $C(x)$ contains a neighborhood of x). According to Equation (32), we must have $\delta(C_{i(x)}) = \delta(C_x) > 0$, proving Equation (30).

It remains to prove Equation (32). Recall that the U_i's are arranged in the decreasing order. Fix $i \geq 1$ and observe that the distribution of $(U_i, \tilde{Y}_i, \eta)$ is invariant under the shift

$$T_x(u, f_1, f_2) = (u, f_1(\cdot + x), f_2(\cdot + x)), \quad u > 0,\ f_1, f_2 \in \mathcal{F}_0.$$

Then we observe that

$$\begin{aligned} \frac{\lambda(\tilde{C}_i \cap B_r)}{\lambda(B_r)} &= \frac{1}{\lambda(B_r)} \int_{B_r} 1_{\{x \in \tilde{C}_i\}} \lambda(\mathrm{d}x) \\ &= \frac{1}{\lambda(B_r)} \int_{B_r} 1_{\{U_i \tilde{Y}_i(x) = \eta(x)\}} \lambda(\mathrm{d}x) \\ &= \frac{1}{\lambda(B_r)} \int_{B_r} 1_{\{T_x(U_i, \tilde{Y}_i, \eta) \in A\}} \lambda(\mathrm{d}x) \end{aligned}$$

with $A = \{(u, f_1, f_2); u f_1(0) = f_2(0)\}$. We can then apply the multi-parameter ergodic theorem (see, e.g., [24, Theorem 2.8]) and conclude that

$$\lim_{r \to +\infty} \frac{\lambda(\tilde{C}_i \cap B_r)}{\lambda(B_r)} = \mathbb{E}[1_A(U_i, \tilde{Y}_i, \eta) \mid \mathcal{I}] \quad \text{a.s.},$$

where $\mathcal{I}$ denotes the σ-algebra of shift-invariant sets. This shows that $\tilde{C}_i$ has an asymptotic density,

$$\delta(\tilde{C}_i) = \mathbb{E}[1_{\{0 \in \tilde{C}_i\}} \mid \mathcal{I}] \quad \text{a.s.}$$

Furthermore, we observe that shift-invariance implies that

$$\mathbb{E}[1_{\{0 \in \tilde{C}_i\}} \mid \mathcal{I}] = \mathbb{E}[1_{\{x \in \tilde{C}_i\}} \mid \mathcal{I}], \quad x \in \mathcal{X}.$$

Using the fact that $\{\delta(\tilde{C}_i) = 0\} \in \mathcal{I}$, we deduce that

$$\begin{aligned} \mathbb{E}[\lambda(\tilde{C}_i) 1_{\{\delta(\tilde{C}_i) = 0\}} \mid \mathcal{I}] &= 1_{\{\delta(\tilde{C}_i) = 0\}} \int_{\mathcal{X}} \mathbb{E}[1_{\{x \in \tilde{C}_i\}} \mid \mathcal{I}]\, \lambda(\mathrm{d}x) \\ &= 0. \end{aligned}$$

Taking the expectation, we obtain that

$$\mathbb{E}[\lambda(\tilde{C}_i)1_{\{\delta(\tilde{C}_i)=0\}}] = 0$$

and we conclude that $\lambda(\tilde{C}_i) = 0$ on the event $\{\delta(\tilde{C}_i) = 0\}$, proving Equation (32). □

□

Proof of Corollary 15. For the sake of brevity, we omit the proof which is quite straightforward from Theorem 14 and very similar to the proof of Corollary 13. □

5.4 Proofs of Propositions 16 and 17

Proof of Proposition 16. The proposition is a reformulation of the criterion for ergodicity/mixing of max-stable processes; see Kabluchko and Schlather [8]. Let $\theta(x, y)$ be the extremal coefficient defined by Equation (7). It is known that η is ergodic if and only if

$$\lim_{r\to+\infty} \frac{1}{\lambda(B_r)} \int_{B_r} (2 - \theta(0, y))\lambda(\mathrm{d}y) = 0, \qquad (33)$$

and that η is mixing if and only if

$$\lim_{y\to\infty} (2 - \theta(0, y)) = 0. \qquad (34)$$

Clearly, in view of Equation (8), Equation (33) is equivalent to

$$\lim_{r\to+\infty} \frac{1}{\lambda(B_r)} \int_{B_r} \mathbb{P}[y \in C(0)]\lambda(\mathrm{d}y) = \lim_{r\to+\infty} \mathbb{E}\Big[\frac{\lambda(C(0) \cap B_r)}{\lambda(B_r)}\Big] = 0$$

and Equation (34) is equivalent to

$$\lim_{y\to\infty} \mathbb{P}[y \in C(0)] = 0.$$

□

Proof of Proposition 17. We use here an upper bound for the β-mixing coefficient provided by Dombry and Eyi-Minko [4, Theorem 3.1]: The β-mixing coefficient $\beta(S_1, S_2)$ is defined by Equation (22) and satisfies

$$\beta(S_1, S_2) \leq 2\mathbb{P}[A(S_1, S_2)],$$

where

$$\begin{aligned} & A(S_1, S_2) \\ = \ & \{\exists i \geq 1,\ \exists (s_1, s_2) \in S_1 \times S_2,\ U_iY_i(s_1) = \eta(s_1) \text{ and } U_iY_i(s_1) = \eta(s_1)\}. \end{aligned}$$

Introducing the cells $C(s_1)$ with $s_1 \in S_1$, we have

$$\begin{aligned} A(S_1, S_2) &= \{\exists (s_1, s_2) \in S_1 \times S_2,\ s_2 \in C(s_1)\} \\ &= \{\cup_{s_1 \in S_1} C(s_1) \cap S_2 \neq \emptyset\} \end{aligned}$$

and

$$\beta(S_1, S_2) \leq 2\mathbb{P}[\cup_{s_1 \in S_1} C(s_1) \cap S_2 \neq \emptyset].$$

For a compact set $K \subset \mathcal{X}$,

$$\beta_r(K) = \beta(K, K_r^c) \leq 2\mathbb{P}[\exists x \in \mathcal{X},\ d(x, K) \geq r \text{ and } x \in \cup_{s \in K} C(s)].$$

We will prove below that if $C(0)$ is bounded a.s., then so is $\cup_{s \in K} C(s)$, whence the right-hand side in the above inequality converges to 0 (by the monotone convergence theorem), and

$$\lim_{r \to \infty} \beta_r(K) = 0.$$

Suppose now that $C(0)$ is bounded a.s. In the discrete case $\mathcal{X} = \mathbb{Z}^d$, the compact set K is finite and $\cup_{s \in K} C(s)$ is a.s. bounded as a finite union of bounded sets. In the continuous case $\mathcal{X} = \mathbb{R}^d$, K may be infinite but it is known that there are a.s. only finitely many indices $i \geq 1$ such that $U_i Y_i(s) = \eta(s)$ for some $s \in K$ (see Dombry and Eyi-Minko [5, Proposition 1]). Hence, we can extract a finite covering $\cup_{s \in K} C(s) = \cup_{j=1}^k C(s_j)$ and $\cup_{s \in K} C(s)$ is a.s. bounded as a finite union of bounded sets. □

A Non-singular flow representation and associated decompositions

In this section we recall some facts on the conservative/dissipative and positive/null decompositions. We also prove some new characterizations of these decompositions. Our approach follows Wang and Stoev [25, section 6] and Wang *et al.* [24]. For more details on non-singular ergodic theory, the reader should refer to Krengel [11].

A.1 Non-singular flow representation

Definition 26. *A measurable non-singular flow on a measure space* $(S, \mathcal{B}, \mu)$ *is a family of functions* $\phi_x : S \to S$, $x \in \mathcal{X}$, *satisfying*

i) (flow property) for all $s \in S$ *and* $x_1, x_2 \in \mathcal{X}$,

$$\phi_0(s) = s \quad \textit{and} \quad \phi_{x_1 + x_2}(s) = \phi_{x_2}(\phi_{x_1}(s));$$

ii) (measurability) the mapping $(x,s) \mapsto \phi_x(s)$ is measurable from $\mathcal{X} \times S$ to S;

iii) (non-singularity) for all $x \in \mathcal{X}$, the measures $\mu \circ \phi_x^{-1}$ and μ are equivalent, i.e. for all $A \in \mathcal{B}$, $\mu(\phi_x^{-1}(A)) = 0$ if and only if $\mu(A) = 0$.

The non-singularity property ensures that one can define the Radon–Nikodym derivative

$$\omega_x(s) = \frac{d(\mu \circ \phi_x^{-1})}{\mathrm{d}\mu}(s). \tag{35}$$

By the measurability property, one may assume that the mapping $(x,s) \mapsto \omega_x(s)$ is jointly measurable on $\mathcal{X} \times S$.

According to de Haan and Pickands [3] and Wang *et al.* [25], any measurable stationary max-stable random field admits a representation of the form

$$\eta(x) = \bigvee_{i \geq 1} U_i f_x(s_i), \quad x \in \mathcal{X}, \tag{36}$$

where $f_x(s) = \omega_x(s) f_0(\phi_x(s))$ and

- $(\phi_x)_{x \in \mathcal{X}}$ is a measurable non-singular flow on some probability space $(S, \mathcal{B}, \mu)$, with $\omega_x(s)$ defined by (35),
- $f_0 \in L^1(S, \mathcal{B}, \mu)$ is nonnegative such that $\int_S f_0 \mathrm{d}\mu = 1$ and the set $\{f_0 = 0\}$ contains no $(\phi_x)_{x \in \mathcal{X}}$-invariant measurable set $B \subset S$ of positive measure,
- $\{(U_i, s_i)\}_{i \geq 1}$ is the enumeration of the points of a Poisson point process on $(0, +\infty) \times S$ with intensity $u^{-2}\mathrm{d}u\mu(\mathrm{d}s)$.

Representation (36) is sometimes written with an extremal integral rather than with a Poisson point process, but the two approaches coincide. Starting with the non-singular flow representation (36), one easily gets a de Haan representation of the form (1) by considering the i.i.d. stochastic processes $Y_i(x) = f_x(s_i)$, $i \geq 1$.

A.2 The conservative/dissipative decomposition

Definition 27. *Consider a measure space $(S, \mathcal{B}, \mu)$ and a measurable non-singular map $\phi : S \to S$. A measurable set $W \subset S$ is said to be wandering if the sets $\phi^{-n}(W)$, $n = 0, 1, 2, \ldots$, are disjoint.*

The Hopf decomposition theorem states that there exists a partition of S into two disjoint measurable sets $S = C \cup D$, $C \cap D = \emptyset$, such that

i) C and D are ϕ-invariant,

ii) there exists no wandering set $W \subset C$ with positive measure,

iii) there exists a wandering set $W_0 \subset D$ such that $D = \cup_{k\in\mathbb{Z}}\phi^k(W_0)$.

This decomposition is unique mod μ and is called the Hopf decomposition of S associated to ϕ; the sets C and D are called the conservative and dissipative components with respect to ϕ, respectively. Given a one-dimensional measurable non-singular flow $(\phi_x)_{x\in\mathcal{X}}$ (with $\mathcal{X} = \mathbb{Z}$ or $\mathbb{R}$), one can consider the Hopf decomposition $S = C_x \cup D_x$ with respect to ϕ_x, for each $x \in \mathcal{X} \setminus \{0\}$. Using measurability, one can show that there exists a decomposition $S = C \cup D$, $C \cap D = \emptyset$, such that $\mu(C_x \Delta C) = \mu(D_x \Delta D) = 0$ for all $x \in \mathcal{X} \setminus \{0\}$ (see Krengel [11] or Rosinsky [15]). This is the conservative/dissipative decomposition of the flow $(\phi_x)_{x\in\mathcal{X}}$. It can be used to define the decomposition $\eta = \eta_C \vee \eta_D$ of the stationary max-stable process η into its conservative and dissipative components

$$\begin{aligned}\eta_C(x) &= \bigvee_{i\geq 1} U_i\omega_x(s_i)f_0(\phi_x(s_i))1_{\{s_i\in C\}},\\ \eta_D(x) &= \bigvee_{i\geq 1} U_i\omega_x(s_i)f_0(\phi_x(s_i))1_{\{s_i\in D\}}.\end{aligned}$$

The processes η_C and η_D are independent and their distribution does not depend on the particular choice of the representation (36). The following simple integral test on the spectral functions allows to retrieve the conservative/dissipative decomposition; see Wang and Stoev [25, Theorem 6.2].

Theorem 28. *We have*

i) $\int_{\mathcal{X}} f_x(s)\lambda(\mathrm{d}x) = \infty$ *for μ-almost all $s \in C$,*

ii) $\int_{\mathcal{X}} f_x(s)\lambda(\mathrm{d}x) < \infty$ *for μ-almost all $s \in D$.*

At this stage, it is not clear why the above decomposition $\eta = \eta_C \vee \eta_D$ based on the conservative/dissipative decomposition $S = C \cup D$ is related to our alternative approach based on the identity $\mathcal{F}_0 = \mathcal{F}_C \cup \mathcal{F}_D$, where the cones $\mathcal{F}_C$ and $\mathcal{F}_D$ were defined in (9) and (10). As we will see, both decompositions do indeed coincide in the sample continuous case so that there is no inconsistency in our notation. The relationship is made through the notion of mixed moving maximum representation defined in the general case $d \geq 1$.

Definition 29. *A stationary max-stable process η is said to have a mixed moving maximum representation (shortly M3-representation) if*

$$\eta(x) \stackrel{d}{=} \bigvee_{i=1}^{\infty} V_i Z_i(x - X_i), \quad x \in \mathcal{X},$$

where

- $\{(V_i, X_i), i \geq 1\}$ *is a Poisson point process on* $(0, +\infty) \times \mathcal{X}$ *with intensity* $u^{-2}\mathrm{d}u\lambda(\mathrm{d}x)$,
- $(Z_i)_{i\geq 1}$ *are i.i.d. copies of a nonnegative measurable stochastic process* Z *on* $\mathcal{X}$ *satisfying* $\mathbb{E}[\int_{\mathcal{X}} Z(x)\lambda(\mathrm{d}x)] = 1$,
- $\{(V_i, X_i), i \geq 1\}$ *and* $(Z_i)_{i\geq 1}$ *are independent.*

Remark 30. Note that Definition 29 implies that

$$\mathbb{P}[\eta(x) \leq u] = \exp\left(-\frac{1}{u}\mathbb{E}\left[\int_{\mathcal{X}} Z(x-y)\lambda(\mathrm{d}y)\right]\right) = \exp(-1/u),$$

so that the margins of η are unit Fréchet. Since η is continuous, we have furthermore for all compact $K \subset \mathcal{X}$,

$$\begin{aligned}\mathbb{P}\left[\max_{x\in K} \eta(x) \leq u\right] &= \exp\left(-\frac{1}{u}\mathbb{E}\left[\int_{\mathcal{X}} \sup_{x\in K} Z(x-y)\lambda(\mathrm{d}y)\right]\right) \\ &= \exp(-\theta(K)/u),\end{aligned}$$

with

$$\theta(K) = \mathbb{E}\left[\int_{\mathcal{X}} \sup_{x\in K} Z(x-y)\lambda(\mathrm{d}y)\right] < \infty. \tag{37}$$

In the case $d = 1$, it is known that a stationary max-stable process η admits a M3-representation if and only if it is purely dissipative, i.e. $\mu(C) = 0$ and $\eta_C = 0$; see Wang and Stoev [25, Theorem 6.4]. Unfortunately, the Hopf decomposition does not extend to multiparameter flows with $d \geq 2$ (*cf.* Krengel [11, page 218]). The following theorem extends the criterion for the existence of a M3-representation to the general case $d \geq 1$.

Theorem 31. *Let* η *be a stationary max-stable process given by the non-singular flow representation* (36). *In the case* $\mathcal{X} = \mathbb{R}^d$, *assume furthermore that* η *has continuous sample paths. Then, the following statements are equivalent:*

i) η *has a M3-representation,*

ii) η *is purely dissipative, i.e.* $\eta_C = 0$ *with* η_C *given by* (11).

Proof. *We prove i)* $\Rightarrow$ *ii).* Note that $\int_{\mathcal{X}} Z(x)\lambda(\mathrm{d}x) < \infty$ a.s. because the expectation of this random variable is required to be 1. In the discrete case $\mathcal{X} = \mathbb{Z}^d$, this immediately implies that $\lim_{x\to\infty} Z(x) = 0$ a.s. In the continuous case $\mathcal{X} = \mathbb{R}^d$, we see from Remark 30 that the continuity of η implies that

$$\int_{\mathcal{X}} \sup_{y\in K} Z(x+y)\lambda(\mathrm{d}x) < \infty \quad \text{a.s.},$$

for all compact sets $K \subset \mathcal{X}$. Hence, $\lim_{x\to\infty} Z(x) \to 0$ a.s. (Otherwise, we can find $x_n \to \infty$ with $Z(x_n) \geq \varepsilon$ and choosing $K = [-1,1]^d$, one can see that $\sup_{y\in K} Z(x+y) \geq \varepsilon$ for all $x \in \cup_{n\geq 1}(x_n + [-1,1]^d)$ so that the integral diverges).

The next step is to go from the mixed moving maximum representation in Definition 29 to the standard de Haan representation (1) by setting

$$Y(x) = \frac{1}{d(X)} Z(x - X),$$

where $d : \mathcal{X} \to (0,\infty)$ is a positive density function, and X is an $\mathcal{X}$-valued random variable which has density d and is independent of Z. Indeed, one checks easily that

$$\left(\bigvee_{i=1}^{\infty} V_i Z_i(x - X_i)\right)_{x\in\mathcal{X}} \stackrel{d}{=} \left(\bigvee_{i=1}^{\infty} U_i Y_i(x)\right)_{x\in\mathcal{X}}.$$

We have seen that $\lim_{x\to\infty} Z(x) = 0$ a.s. and this clearly implies that $\lim_{x\to\infty} Y(x) = 0$. We deduce that $Y \in \mathcal{F}_D$ a.s. and $\eta_C = 0$. Hence, η is purely dissipative.

We prove ii) $\Rightarrow$ i). Consider a measurable max-stable process η with no conservative component, i.e. η has a representation of the form $\eta = \vee_{i\geq 1} U_i Y_i$ with $Y_i \in \mathcal{F}_D$ almost surely. We want to show that η admits an M3-representation. The proof is very similar to the proof of Theorem 14 in Kabluchko *et al.* [9] and we sketch only the main lines. Thanks to the condition $\lim_{x\to\infty} Y_i(x) = 0$, we can consider the random variables

$$\begin{aligned} X_i &= \arg\max_{x\in\mathcal{X}} Y(x), \\ Z_i(\cdot) &= \frac{Y_i(X_i + \cdot)}{\max_{x\in\mathcal{X}} Y_i(x)}, \\ V_i &= U_i \max_{x\in\mathcal{X}} Y_i(x), \end{aligned} \tag{38}$$

where the arg max is the point $x \in \mathcal{X}$ achieving the maximum which is the smallest with respect to the lexicographic order. Clearly, we have $U_i Y_i(x) = V_i Z_i(x - X_i)$ for all $x \in \mathcal{X}$ so that

$$\eta(x) = \bigvee_{i\geq 1} V_i Z_i(x - X_i).$$

It remains to check that $(V_i, X_i, Z_i)_{i\geq 1}$ has the properties required in Definition 29, i.e. is a Poisson point process with product intensity $u^{-2}\mathrm{d}u\lambda(\mathrm{d}x)Q(\mathrm{d}f)$, where Q a probability measure on $\mathcal{F}_0$. Clearly, $(V_i, X_i, Z_i)_{i\geq 1}$ is a Poisson point process as the image of the original point process $(U_i, Y_i)_{i\geq 1}$. Its intensity is the image of the intensity of the original point process. With a straightforward transposition of the arguments of [9, Theorem 14], one can check that it has the required form. $\square$

Example 32. The assumption that the sample paths of η should be continuous cannot be removed from Theorem 31. To see this, consider the (deterministic) process Z of the following form:

$$Z(x) = C \sum_{n=1}^{\infty} f(n^2(x-n)), \quad x \in \mathbb{R},$$

where $f(t) = (1-t^2)1_{|t|\leq 1}$ and C is a constant such that $\int_{\mathbb{R}} Z(x)\mathrm{d}x = 1$. The process Z is non-zero only on the intervals of the form $(n-\frac{1}{n^2}, n+\frac{1}{n^2})$. The M3-process η corresponding to Z is well-defined. On the other hand, $\mathbb{P}[Z \in \mathcal{F}_D] = 0$ and hence, $\mathbb{P}[Y \in \mathcal{F}_D] = 0$, where Y is the spectral function of η from the de Haan representation (1). It is easy to check that (37) fails meaning that the sample paths of η are a.s. not locally bounded and hence not continuous.

Theorem 31 has an interesting generalization based on the Definition 23 of localizable cone.

Theorem 33. *Let η be a measurable max-stable process given by the representation* (1). *Assume that there is a localizable cone $\mathcal{F}_L$ such that $\mathbb{P}[Y \in \mathcal{F}_L] = 1$. Then η has a M3-representation.*

Proof. According to Theorem 31, the result holds in the particular case of the localizable cone $\mathcal{F}_D = \{f \in \mathcal{F}_0; \lim_\infty f = 0\}$. The proof in the general case of a localizable cone is exactly the same replacing Equation (38) by

$$X_i = L_1(Y_i), \quad Z_i(\cdot) = \frac{Y_i(X_i + \cdot)}{L_2(Y_i)}, \quad V_i = U_i L_2(Y_i). \tag{39}$$

The invariance properties of L_1 and L_2 ensure that the arguments of the proof of [9, Theorem 14] still work. $\square$

We conclude this subsection with the proofs of Lemmas 22 and 24.

Proof of Lemma 22. As we have seen in the proof $i) \Rightarrow ii)$ of Theorem 31, the inclusion

$$\left\{ \int_{\mathcal{X}} \sup_{y\in K} Y(x+y)\lambda(\mathrm{d}x) < \infty \right\} \subset \left\{ \lim_{x\to\infty} Y(x) = 0 \right\}$$

is trivial. For the reverse inclusion, Theorem 31 implies that $Y1_{\{Y\in\mathcal{F}_D\}}$ admits an M3-representation so that

$$\int_{\mathcal{X}} \sup_{y\in K} Y(x+y)1_{\{Y\in\mathcal{F}_D\}}\lambda(\mathrm{d}x) < \infty \quad \text{a.s.}$$

This implies that (modulo null sets)

$$\left\{\lim_{x\to\infty} Y(x) = 0\right\} \subset \left\{\int_{\mathcal{X}} \sup_{y\in K} Y(x+y)\lambda(\mathrm{d}x) < \infty\right\}.$$

□

Proof of Lemma 24. Theorem 33 implies that $Y1_{\{Y\in\mathcal{F}_L\}}$ admits an M3-representation. Then, Theorem 31 implies that $Y1_{\{Y\in\mathcal{F}_L\}} \in \mathcal{F}_D$ almost surely. The inclusion $\{Y\in\mathcal{F}_L\} \subset \{Y\in\mathcal{F}_D\}$ modulo null sets follows directly. □

A.3 The positive/null decomposition

Definition 34. *Consider a measure space $(S,\mathcal{B},\mu)$ and a measurable non-singular flow $(\phi_x)_{x\in\mathcal{X}}$ on S. A measurable set $W\subset S$ is said to be weakly wandering with respect to $(\phi_x)_{x\in\mathcal{X}}$ if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}\subset\mathcal{X}$ such that $\phi_{x_n}^{-1}(W)\cap\phi_{x_m}^{-1}(W)=\emptyset$ for all $n\neq m$.*

According to Wang *et al.* [24, Theorem 2.4], there exists a partition of S into two disjoint sets $S = P\cup N$, $P\cap N=\emptyset$, such that

i) P and N are ϕ_x-invariant for all $x\in\mathcal{X}$,

ii) P has no weakly wandering set of positive measure,

iii) N is a union of weakly wandering sets.

This decomposition is unique mod μ and is called the Neveu decomposition of S associated to $(\phi_x)_{x\in\mathcal{X}}$; P and N are called the positive and null components with respect to $(\phi_x)_{x\in\mathcal{X}}$, respectively. It can be shown that P is the largest subset of S supporting a finite measure which is equivalent to μ and invariant under the flow $(\phi_x)_{x\in\mathcal{X}}$ ([24, Lemma 2.2]). Hence, there exists a finite measure which is equivalent to μ and invariant under the flow if and only if $N=\emptyset$ mod μ.
The corresponding decomposition of η into its positive and null components is given by $\eta=\eta_P\vee\eta_N$ with

$$\begin{aligned}\eta_P(x) &= \bigvee_{i\geq 1} U_i\omega_x(s_i)f_0(\phi_x(s_i))1_{\{s_i\in P\}},\\ \eta_N(x) &= \bigvee_{i\geq 1} U_i\omega_x(s_i)f_0(\phi_x(s_i))1_{\{s_i\in N\}}.\end{aligned}$$

Interestingly, η_P and η_N are independent and their distribution does not depend on the particular choice of the representation (36). The importance of this decomposition comes from the following theorem (see [7, Theorem 8] and [24, Theorem 5.3]).

Theorem 35. *Let η be a stationary max-stable process given by the non-singular flow representation* (36)*. Then, η is ergodic if and only if its positive component is trivial, i.e. $\eta_P = 0$.*

In the one-dimensional case, an alternative characterization of the positive/null decomposition is known (see Samorodnitsky [18] or Wang and Stoev [25, Theorem 6.3]). It is similar but more involved than Theorem 28 for the conservative/dissipative decomposition. We provide here an alternative result that is simpler and valid in all dimensions $d \geq 1$.

Theorem 36. *Let η be a stationary max-stable process given by the non-singular flow representation* (36)*. For $r > 0$, write $B_r = [-r, r]^d \cap \mathcal{X}$. We have*

i) $\lim_{r\to\infty} \frac{1}{\lambda(B_r)} \int_{B_r} f_x(s)\lambda(\mathrm{d}x)$ *exists and is positive for μ-almost all $s \in P$,*

ii) $\liminf_{r\to\infty} \frac{1}{\lambda(B_r)} \int_{B_r} f_x(s)\lambda(\mathrm{d}x) = 0$ *for μ-almost all $s \in N$.*

Proof. We consider the positive case and the null case separately. Assume first that η is generated by a positive flow. Then, with a possible change of representation, one can assume that μ is a probability measure invariant under the flow. This implies $w_x(s) \equiv 1$ and $f_x(s) = f_0(\phi_x(s))$. By the multiparameter Birkhoff Theorem ([24, Theorem 2.8]), we have

$$\lim_{r\to\infty} \frac{1}{\lambda(B_r)} \int_{B_r} f_x(s)\lambda(\mathrm{d}x) = \mathbb{E}_\mu[f_0 \mid \mathcal{I}] \quad \mu - \text{almost surely},$$

where $\mathcal{I}$ is the σ-algebra of $(\phi_x)_{x\in\mathcal{X}}$-invariant measurable sets. The set $B = \{\mathbb{E}_\mu[f_0 \mid \mathcal{I}] = 0\}$ is measurable and $(\phi_x)_{x\in\mathcal{X}}$-invariant and f_0 vanishes on B (recall f_0 is nonnegative). This implies that $\mu(B) = 0$ (see point ii) in representation (36)). In other terms, the function $x \mapsto f_x(s)$ belongs to the cone $\mathcal{F}_P$ defined by (16) almost surely. This must also be true for any equivalent representation (36) and the result follows in the positive case.

We consider now the case when η is generated by a null flow. One can assume without loss of generality that μ is a probability measure. According to the stochastic ergodic theorem for nonsingular actions ([24, Theorem 2.7]), we have

$$\frac{1}{\lambda(B_r)} \int_{B_r} f_x(\cdot)\lambda(\mathrm{d}x) \overset{\mu}{\to} \tilde{f}_0 \quad \text{as } r \to \infty$$

where $\xrightarrow{\mu}$ denotes convergence in μ-probability and $\tilde{f}_0 \in L^1(S,\mu)$ satisfies

$$\omega_x(\cdot)\tilde{f}_0 \circ \phi_x(\cdot) = \tilde{f}_0 \quad \text{for all } x \in \mathcal{X}.$$

This relation implies that the measure $\tilde{f}_0(s)\mu(\mathrm{d}s)$ is a finite measure which is absolutely continuous with respect to μ and invariant under the flow $(\phi_x)_{x\in\mathcal{X}}$. Since the flow has no positive component, this implies that $\tilde{f}_0 = 0$. We deduce that $\frac{1}{\lambda(B_r)}\int_{B_r} f_x(\cdot)\lambda(\mathrm{d}x)$ converges in μ-probability to 0 and hence almost surely to 0 along a subsequence, whence

$$\liminf_{r\to\infty} \frac{1}{\lambda(B_r)} \int_{B_r} f_x(s)\lambda(\mathrm{d}x) = 0 \quad \mu-\text{almost surely.}$$

□

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